CE 228N: Introduction to the Theory of Plasticity: Homework IV

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1. Use any suitable tool (e.g. MATLAB) to visualize the Drucker-Prager failure / yield surface in principal stress space for a given k and α. It should be clearly labeled, with all axes shown. Remember that $0 < \alpha < 1/\sqrt{3}$. What is the yield strength in pure shear using the Drucker-Prager criterion?

2. Re-derive the result that for Drucker-Prager associated flow, the plastic dissipation is

$$D_p = \frac{k\sqrt{2\dot{\varepsilon}_{ij}^p \dot{\varepsilon}_{ij}^p}}{\sqrt{1+6\alpha^2}}$$

   What is D_p in terms of the friction angle?

3. Consider the Mohr-Coulomb criterion $\tau = c - \mu\sigma$. Re-derive the result that in principal stress space this criterion can be written as

$$\sigma_{max} - \sigma_{min} + (\sigma_{max} + \sigma_{min}) \sin\phi = 2c\cos\phi$$

   in terms of the maximum and minimum principal stresses, where $\tan\phi = \mu$.

4. Sketch the Mohr-Coulomb yield locus in the π-plane and show that it is an irregular hexagon.

5. (Guided problem) Find the stresses in an *elastic* thin-walled cylindrical tube of mean radius c and thickness $t = 2h$ (where $t << c$) subjected to a torque T using the Prandtl

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torsion function approach. Assume that the z-axis is the cylinder axis and that the twisting is also about this axis. The material has shear modulus μ. Proceed as follows:

(a) Use the torsion function method to first solve the problem for a thick-walled cylindrical tube with inner radius a and outer radius b. Recall that for this problem the torsion function φ must satisfy $\nabla^2\varphi = -2\mu\beta$, $\varphi = C$ on $r = a$ and $\varphi = 0$ on $r = b$, where C is a constant to be determined. It is easy to find a simple quadratic function of x and y for this purpose.

(b) The torque-twist relation for this multiply-connected section can be obtained from

$$T = \iint_{\Omega} \left(x\sigma_{yz} - y\sigma_{xz}\right)\, dxdy$$

Use the auxiliary equations

$$\sigma_{xz} = \frac{\partial\varphi}{\partial y} \qquad \sigma_{yz} = -\frac{\partial\varphi}{\partial x}$$

Hence show that

$$T = \frac{\pi\mu\beta}{2}\left(b^4 - a^4\right)$$

where β is the twist per unit length.

(c) Consider the special case of a thin-walled tube by putting $a = c - t/2$, $b = c + t/2$ and use the fact that $t << a, b$. Show that one thus recovers the strength of materials solution

$$\beta = \frac{T}{2\pi\mu t c^3}$$

(d) Find the stresses σ_{xz} and σ_{yz} in the thin-walled tube. Make a sketch of the tube, loading axis, and draw a stress cube showing the non-zero stress components.

(e) Transform the stresses to cylindrical polar coordinates with the z axis as the z-axis of the polar system. Show that only the $\sigma_{\theta z}$ component is non-zero in this case and that it is given by

$$\tau = \sigma_{\theta z} = \frac{T}{2\pi c^2 t}$$

thus establishing the solution used in class.

(f) Write down the matrix of polar stress components when this tube is additionally subjected to an axial tension σ.

(g) For the thick-walled tube problem, take the limit $a \to 0$. Thus show that the shear stress τ in any thin annulus of a solid cylinder in torsion is proportional to the radius r of the annulus.

6. Solve the problem of a thin-walled tube which is twisted to the point of yield, and then stretched. Assume linear hardening with modulus H. This is identical to the problem solved in class, except for the hardening.

7. Use the expression for the grad operator in spherical in the handout to write down the first equation of equilibrium (the r-component equation) in a spherical basis.

   You will need to use $\vec{\boldsymbol{\nabla}} \cdot \boldsymbol{\sigma} = \mathbf{0}$ and $\boldsymbol{\sigma} = \sigma'_{ij}\, \hat{\mathbf{e}}'_i \otimes \hat{\mathbf{e}}'_j$ where the $\hat{\mathbf{e}}'_i$ are the spherical basis vectors and the σ'_{ij} the stress components in spherical coordinates.

8. For the pressurized spherical shell problem, work through and re-derive the solution obtained in class. Show that there is a limiting wall-thickness ratio b/a of about 1.70 below which secondary yielding during unloading is not possible in the spherical shell.

9. An elastic-plastic cylindrical shell of inner radius a and outer radius b is subjected to an internal pressure p. Assume a state of plane strain. Perform an analysis along the lines of what was done in class for the spherical shell, using the Tresca yield criterion, and find

   (a) The elastic stress distribution in the shell (you could use Airy stress functions)

   (b) An expression for p_e, the pressure at first yield

   (c) The stress distribution in the shell when $p > p_e$

   (d) The maximum pressure p^{MAX} that the shell can withstand

   (e) The in-plane displacement field in the cylindrical shell

   (f) A relation between the radius c of the elastic-plastic boundary and the pressure p

   (g) The residual stresses in the cylindrical shell on complete unloading

   (h) If there is a possibility of secondary yielding and under what conditions?

   Assume the material is elastic / perfectly-plastic, with a shear yield stress of k. The elastic moduli of the material are E and ν. State all assumptions clearly and show all your work. Do not forget the role of the σ_{zz} stress component in yielding in plane strain.

10. In Q9 above, plot the normalized radial and circumferential stresses $\sigma_{rr}/2k$ and $\sigma_{\theta\theta}/2k$ against r/b for a few different values of c/b for a shell with $b/a = 2$ given $p > p_e$. Here c is the radius of the elastic-plastic boundary.